Permutation groups with metrizable universal minimal flow

Andy Zucker
Carnegie Mellon University

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When Topological Dynamics Meets Model Theory
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Recall that the closed subgroups of S_∞ are exactly the automorphism groups of relational *Fraïssé structures*.

If $\mathbf{K}$ is a Fraïssé structure, then $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ is a Fraïssé class. Conversely, if $\mathcal{K}$ is a Fraïssé class, there is up to isomorphism a unique Fraïssé structure $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ with $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$.

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It is a fact that for any topological group G, there is up to G-flow isomorphism a unique flow $M(G)$ which is minimal and universal. $M(G)$ is called the *universal minimal flow*.

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Let $\mathcal{K}$ be a class of finite structures, and let $\mathbf{A} \in \mathcal{K}$. We say that **A** is a *Ramsey object* if for every $\mathbf{B} \in \mathcal{K}$ with $\mathbf{B} \geq \mathbf{A}$ and every $k \in \mathbb{N}$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{C} \geq \mathbf{B}$ for which we have

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This says that for every coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \rightarrow [k]$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ so that $|\gamma(f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}))| = 1$.

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We say that $\mathcal{K}$ has the *Ramsey Property* if each $\mathbf{A} \in \mathcal{K}$ is a Ramsey object. We can now state the following theorem.

Theorem (Kechris-Pestov-Todorčević)

Let $\mathbf{K}$ *be a Fraïssé structure,* $\mathcal{K} = \mathrm{Age}(\mathbf{K})$*, and* $G = \mathrm{Aut}(\mathbf{K})$*. Then* $\mathcal{K}$ *has the Ramsey Property iff* G *is extremely amenable (i.e.* $M(G)$ *is a singleton).*

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Let $\mathbf{K}$ *be a Fraïssé structure,* $\mathcal{K} = \mathrm{Age}(\mathbf{K})$*, and* $G = \mathrm{Aut}(\mathbf{K})$*. Then* $\mathcal{K}$ *has the Ramsey Property iff* G *is extremely amenable (i.e.* $M(G)$ *is a singleton).*

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Theorem (Z.)

Let $\mathbf{K}$ be a Fraïssé structure, $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, and $G = \mathrm{Aut}(\mathbf{K})$. Then $M(G)$ is metrizable iff each $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree.

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- For $\mathbf{V}_\infty$ the infinite dimensional vector space over F_q, $M(\mathrm{Aut}(\mathbf{V}_\infty))$ is the space of lex. orderings of $\mathbf{V}_\infty$.

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- For the tournament $\mathbf{S}(2)$, $M(\mathrm{Aut}(\mathbf{S}(2)))$ is the space of admissible labelled 2-part partitions of $\mathbf{S}(2)$.

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Let $\mathcal{K}$ be a Fraïssé class in a language L with limit $\mathbf{K}$. Let $\mathcal{K}^*$ be a Fraïssé class in $L^* = L \cup \{S_i : i \in I\}$, where the S_i are countably many new relation symbols of arity $n(i)$, with limit $\mathbf{K}^*$ and with the property that $\mathbf{K}^*|_L = \mathbf{K}$ (i.e. $\mathcal{K}^*$ is a *reasonable* expansion of $\mathcal{K}$).

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The topological space $X_{\mathcal{K}^*}$ is the collection of all structures of the form $\langle \mathbf{K}, \vec{S}^{\mathbf{K}} \rangle$. If $\mathbf{A} \subseteq \mathbf{K}$, $\mathbf{A} \in \mathcal{K}$, and $\mathbf{A}^* \in \mathcal{K}^*$ with $\mathbf{A}^*|_L = \mathbf{A}$, then this determines a basic open neighborhood of $X_{\mathcal{K}^*}$ via

$$N(\mathbf{A}^*) = \{\vec{S}^{\mathbf{K}} \in X_{\mathcal{K}^*} : \langle \mathbf{A}, \vec{S}^{\mathbf{K}}|_{\mathbf{A}} \rangle = \mathbf{A}^*\}$$

$X_{\mathcal{K}^*}$ is compact iff for each $\mathbf{A} \in \mathcal{K}$, $\{\mathbf{A}^* \in \mathcal{K}^* : \mathbf{A}^*|_L = \mathbf{A}\}$ is finite (i.e. $\mathcal{K}^*$ is *precompact*). $G = \mathrm{Aut}(\mathbf{K})$ acts on $X_{\mathcal{K}^*}$ via the *logic action*, i.e. for $\mathbf{K}' \in X_{\mathcal{K}^*}$, $g \in G$, and each $i \in I$, we have

$$S_i^{\mathbf{K}'\cdot g}(x_1, ..., x_{n(i)}) = S_i^{\mathbf{K}'}(g(x_1), ..., g(x_{n(i)}))$$

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We say that $\mathcal{K}^*$ has the *Expansion Property* if for any $\mathbf{A} \in \mathcal{K}$, there is $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$ so that for any expansions $\mathbf{A}^*$, $\mathbf{B}^*$ of $\mathbf{A}$ and $\mathbf{B}$ respectively, we have $\mathbf{A}^* \leq \mathbf{B}^*$.

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Theorem (Kechris-Pestov-Todorčević, Nguyen Van Thé)

Let $\mathbf{K}$ be a Fraïssé structure, $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, and $G = \mathrm{Aut}(\mathbf{K})$. Let $\mathcal{K}^*$ be a reasonable, precompact Fraïssé expansion of $\mathcal{K}$. Then $M(G) \cong X_{\mathcal{K}^*}$ iff $\mathcal{K}^*$ has the ExpP and the RP.

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If G is a closed subgroup of S_∞ with $M(G)$ metrizable, can $M(G)$ be described using a logic action as above?

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Yes!

Theorem (Z.)

Let $\mathbf{K}$ be a Fraïssé structure, $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, and $G = \mathrm{Aut}(\mathbf{K})$. Suppose $M(G)$ is metrizable. Then $\mathcal{K}$ admits a reasonable, precompact Fraïssé expansion class $\mathcal{K}^*$ with the Expansion Property and the Ramsey Property.

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If G is a topological group and X is a minimal G-flow, then $x \in X$ is a *generic point* if $x \cdot G$ is comeager. G is said to have the *Generic Point Property* if each minimal flow has a generic point. This holds iff $M(G)$ has a generic point.

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If G, $\mathcal{K}$, and **K** are as always and $\mathcal{K}^*$ is a reasonable Fraïssé expansion of $\mathcal{K}$ with the Expansion Property, then the orbit of $\mathbf{K}^* = \mathrm{Flim}(\mathcal{K}^*)$ is generic.

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Corollary (Z.)

Let G be a closed subgroup of S_∞, and suppose $M(G)$ is metrizable. Then G has the Generic Point Property.

However, the Generic Point Problem as originally asked is still open.

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Problem (Angel, Kechris, Lyons)

Let G be a Polish group, and suppose $M(G)$ is metrizable. Then does $M(G)$ have the Generic Point Property?

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Let $\mathbf{D}$ be a countably infinite relational structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and let $\mathbf{A} \in \mathcal{D}$. We say $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *thick* if for every $\mathbf{B} \in \mathbf{D}$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq T$.

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We consider partial colorings $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \to [k]$; we say γ is *full* if $\mathrm{dom}(\gamma) = \mathrm{Emb}(\mathbf{A}, \mathbf{D})$, and we say γ is *large* if $\mathrm{dom}(\gamma)$ is thick.

Proposition

Suppose **D** *is a countably infinite relational structure,* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, *and* $\mathcal{C}$ *is cofinal in* $\mathcal{D}$. *Let* $\mathbf{A} \in \mathcal{C}$ *and fix any* $k \geq 2$. *Then the following are equivalent:*

1. **A** *is a Ramsey object in* $\mathcal{C}$,
2. **A** *is a Ramsey object in* $\mathcal{D}$,
3. *For any full* k*-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$, *there is some* γ_i *which is thick,*
4. *For any large* k*-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$, *there is some* γ_i *which is thick.*

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Proposition

Suppose $\mathbf{D}$ *is a countably infinite relational structure,* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, *and* $\mathcal{C}$ *is cofinal in* $\mathcal{D}$. *Let* $\mathbf{A} \in \mathcal{C}$ *and fix any* $r > k$. *Then the following are equivalent:*

1. $\mathbf{A}$ *has Ramsey degree* $t \leq k$ *in* $\mathcal{C}$,
2. $\mathbf{A}$ *has Ramsey degree* $t \leq k$ *in* $\mathcal{D}$,
3. *Any full* r*-coloring of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *has some subset of* k *colors which form a thick subset,*
4. *Any large* r*-coloring of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *has some subset of* k *colors which form a thick subset.*

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With $\mathbf{D}$, $\mathcal{D}$ as above and $\mathbf{A} \in \mathcal{D}$, we say that $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *syndetic* if $\mathrm{Emb}(\mathbf{A}, \mathbf{D}) \setminus S$ is not thick.

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If γ is a full k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$, we say that γ is syndetic if each γ_i is syndetic.

Proposition

With $\mathbf{D}$, $\mathcal{D}$, and $\mathbf{A}$ as above, then $\mathbf{A}$ has Ramsey degree $t \geq k$ (t possibly infinite) iff there is a syndetic k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$.

Let $\mathbf{A}, \mathbf{B} \in \mathcal{D}$ with $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. We define $\hat{f} : \mathrm{Emb}(\mathbf{B}, \mathbf{D}) \to \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ via $\hat{f}(x) = x \circ f$.

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We often consider these "dual" maps when dealing with a Fraïssé structure $\mathbf{K}$ with age $\mathcal{K}$. Notice that $\mathbf{K}$ is a Fraïssé structure iff every such $\hat{f}$ is surjective.

Using the amalgamation property, we obtain the following:

Let $\mathbf{A}, \mathbf{B} \in \mathcal{D}$ with $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. We define $\hat{f} : \mathrm{Emb}(\mathbf{B}, \mathbf{D}) \to \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ via $\hat{f}(x) = x \circ f$.

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Proposition

Let $\mathbf{K}, \mathcal{K}$ be as above, and fix $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. Then $X \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is thick (resp. syndetic) iff $\hat{f}^{-1}(X) \subseteq \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ is thick (resp. syndetic).

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Corollary

Let $\mathbf{K}, \mathcal{K}, \mathbf{A} \leq \mathbf{B}$ be as above. Then if $\mathbf{B}$ has Ramsey degree k, then $\mathbf{A}$ has Ramsey degree $t \leq k$. In particular, if $\mathbf{B}$ is a Ramsey object, then so is $\mathbf{A}$.

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This is not in general true for the "substructure" version of the Ramsey property.

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If (X, x_0) and (Y, y_0) are G-ambits, then $f : X \to Y$ is a *map of G-ambits* if f is a G-map sending x_0 to y_0. There is at most one map of ambits from (X, x_0) to (Y, y_0); if there is one, we write $(X, x_0) \geq (Y, y_0)$.

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It is a fact that every topological group G admits up to isomorphism a unique *greatest ambit* $(S(G), 1)$; any minimal subflow of $S(G)$ is universal, hence isomorphic to $M(G)$.

From now on, we fix once and for all a Fraïssé structure $\mathbf{K}$ with age $\mathcal{K}$. We also set $G = \mathrm{Aut}(\mathbf{K})$. Fix finite substructures $\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \cdots$ with $\mathbf{K} = \bigcup_n \mathbf{A}_n$. Write $H_n = \mathrm{Emb}(\mathbf{A}_n, \mathbf{K})$.

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For $m \leq n$, let $i_m^n : \mathbf{A}_m \hookrightarrow \mathbf{A}_n$ be the inclusion map. This gives rise to a surjective dual map $\hat{\imath}_m^n : H_n \to H_m$. Note that if $m \leq n \leq N$, then $\hat{\imath}_n^N = \hat{\imath}_m^n \circ \hat{\imath}_n^N$.

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Form βH_n, the space of all ultrafilters on H_n. Each $\hat{\imath}_m^n$ extends to a continuous surjective $\tilde{\imath}_m^n : \beta H_n \to \beta H_m$. If $p \in \beta H_n$ and $S \subseteq H_m$, then $S \in \tilde{\imath}_m^n$ iff $(\hat{\imath}_m^n)^{-1}(S) \in p$.

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Now form the inverse limit $\varprojlim \beta H_n$ along the maps $\tilde{\imath}_m^n$. A basic open neighborhood of $\alpha \in \varprojlim \beta H_n$ is given by $\{\alpha' \in \varprojlim \beta H_n : S \in \alpha'(m)\}$ for some $m \in \mathbb{N}$ and $S \subseteq H_m$, $S \in \alpha(m)$.

G acts on $\varprojlim \beta H_n$ as follows: if $\alpha \in \varprojlim \beta H_n$, $g \in G$, and $S \in H_m$, then $S \in \alpha g(m)$ iff for some $n \geq m$, $\{x \in H_n : x \circ g|_m \in S\} \in \alpha(n)$. This action is jointly continuous!

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Theorem (Pestov)

$(\varprojlim \beta H_n, 1)$ *is the greatest G-ambit.*

We can now give a new proof of the extreme amenability result from KPT. In fact, the proofs of many of the other results work by mimicking the methods in the proof I will present here, so this proof is in some ways representative.

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$R_n \neq \emptyset$ iff $\mathbf{A}_n$ is a Ramsey object.

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If $m \leq n$, $\mathbf{A}_n$ is a Ramsey object, and $p \in R_m$, then there is $q \in R_n$ with $\tilde{\imath}^n_m(q) = p$.

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Theorem

$\alpha \in \varprojlim \beta H_n$ is a fixed point iff $\alpha \in \varprojlim R_n$. In particular, G is extremely amenable iff $\mathcal{K}$ has the Ramsey Property.

Let $\alpha \in \varprojlim R_n$; suppose for sake of contradiction that there were some $g \in G$ with $\alpha g \neq \alpha$. In particular, there is $m \in \mathbb{N}$ and $S \subseteq H_m$, $S \in \alpha(m)$ with $S \notin \alpha g(m)$.

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For some $n \geq m$, we have $T_1 := \{x \in H_n : x \circ i_m^n \in S\} \in \alpha(n)$ and $T_2 := \{x \in H_n : x \circ g|_m \notin S\} \in \alpha(n)$. So $T_1 \cap T_2 \in \alpha(n)$.

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Now for large $N \geq n$, find $h \in H_N$ with $h \circ \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_N) \subseteq T_1 \cap T_2$. Now consider $h \circ g|_n \circ i_m^n = h \circ i_n^N \circ g|_m$. The left side is in S, but the right side is not, a contradiction.

Suppose $\alpha(m)$ is not thick, with $S \in \alpha(m)$ not thick. Find $n \geq m$ with $f \circ \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n) \not\subseteq S$ for every $f \in H_n$.

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Pick r with $T_r \notin \alpha(n)$. Then for any $g \in G$ with $g|_m = r$, then $\alpha g \neq \alpha$.

Thanks!